SOLUTION OF LARGE UNDERDETERMINED LINEAR SYSTEMS FOR A GENERALIZED NON-HOMOGENEOUS NETWORK FLOW PROGRAMMING PROBLEM

L.A. PILIPCHUK AND E.S. VECHARYNSKI

Abstract. We construct a general solution for a system of linear equations corresponding to the system of main constraints for a broad class of generalized non-homogeneous network flow programming problems.

1. Introduction

Let $S = (I, U)$ be a finite oriented connected network without multiple arcs and loops, where I is a set of nodes and U is a set of arcs, $U \subset I \times I (|I| < \infty, |U| < \infty)$. Let K $(|K| < \infty)$ be a set of different types of flow transported through the network S. We assume that $K = \{1, \dots, |K|\}$. Let us denote a connected network corresponding to a certain type of flow $k \in K$ with $S^k = (I^k, U^k), I^k \subseteq I, U^k = \{(i,j)^k : (i,j) \in \widetilde{U}^k\}, \widetilde{U}^k \subseteq U$ - a set of arcs of the network S carrying the flow of type k. Also, we define sets $K(i) = \{k \in K : i \in I^k\}$ and $K(i,j) = \{k \in K : (i,j)^k \in U^k\}$ of types of flow transported through a node $i \in I$ and an arc $(i,j) \in U$ respectively. Let us introduce a subset U_0 of the set U, and let $K_0(i,j) \subseteq K(i,j), (i,j) \in U_0$ be an arbitrary subset of $K(i,j)$ such that $|K_0(i,j)| > 1$.

Consider the following linear underdetermined system

$$\sum_{j \in I_i^+(U^k)} x_{ij}^k - \sum_{j \in I_i^-(U^k)} \mu_{ji}^k x_{ji}^k = a_i^k, \quad i \in I^k,\, k \in K, \tag{1.1}$$

$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} \lambda_{ij}^{kp} x_{ij}^k = \alpha_p,\, p = \overline{1,q}, \tag{1.2}$$

$$\sum_{k \in K_0(i,j)} x_{ij}^k = z_{ij}, \quad (i,j) \in U_0, \tag{1.3}$$

where $I_i^+(U^k) = \{j \in I^k : (i,j)^k \in U^k\}$, $I_i^-(U^k) = \{j \in I^k : (j,i)^k \in U^k\}$; $a_i^k, \lambda_{ij}^{kp}, \alpha_p, z_{ij} \in \mathbb{R}, \mu_{ij}^k > 0$ - parameters of the system; $x = (x_{ij}^k, (i,j)^k \in U^k, k \in K)$- vector of unknowns. We assume that $\sum_{k \in K} |I^k| + q + |U_0| < \sum_{k \in K} |U^k|$ and rank of the system (1.1) - (1.3) is equal to $\sum_{k \in K} |I^k| + q + |U_0|$. Further, we will call (1.1) the network part, and (1.2) - (1.3) - the additional part, of the system (1.1) - (1.3).

2. Solution of the network part of the system

We split the solution of the system (1.1) into solutions of $|K|$ systems corresponding to a fixed $k \in K$.

Let $U_L = \{U_L^k \subseteq U^k, k \in K\}$ be a support of the network $S = (I, U)$ for system (1.1) [1, 2]. Recall, a cycle $L^k \subseteq S^k$ is called non-singular [1, 3] if $\prod_{(i,j)^k \in L^{k+}} \mu_{ij}^k \neq \prod_{(i,j)^k \in L^{k-}} \mu_{ij}^k$, where L^{k+}, L^{k-} - sets of forward and backward arcs respectively.

Theorem 1. *(Network support criterion.) The set $U_L = \{U_L^k, k \in K\}$ is a support of the network $S = (I, U)$ for system (1.1) iff for each $k \in K$ the network $S_L^k = (I^k, U_L^k)$ is a union $S_L^k = \bigcup_t S_L^{k,t}$ of connectivity components $S_L^{k,t} = (I(U_L^{k,t}), U_L^{k,t})$, each containing a unique non-singular cycle, $U_L^k = \bigcup_t U_L^{k,t}, I^k = \bigcup_t I(U_L^{k,t})$.*

Let us introduce a characterictic vector $\delta^k(\tau, \rho) = (\delta_{ij}^k(\tau, \rho), (i, j)^k \in U^k)$, where $k \in K$ is fixed, entailed by an arc $(\tau, \rho)^k \in U^k \setminus U_L^k$ with respect to the support U_L^k, as a solution vector of the following system:

$$\sum_{j \in I_i^+(B_{\tau\rho}^k)} \delta_{ij}^k(\tau, \rho) - \sum_{j \in I_i^-(B_{\tau\rho}^k)} \mu_{ji}^k \delta_{ji}^k(\tau, \rho) = 0, \quad i \in I^k, B_{\tau\rho}^k = U_L^k \cup (\tau, \rho)^k, \tag{2.1}$$

$$\delta_{\tau\rho}^k(\tau, \rho) = 1, \delta_{ij}^k(\tau, \rho) = 0, \quad (i, j)^k \in U^k \setminus (U_L^k \cup (\tau, \rho)^k). \tag{2.2}$$

The system of characteristic vectors, entailed by (all) different arcs $(\tau, \rho)^k \in U^k \setminus U_L^k$ is a basis of a solution space of the homogeneous system corresponding to (1.1), where $k \in K$ is fixed. Thus, for a fixed $k \in K$,we represent solutions of the system (1.1) as a sum of a general solution of the corresponding homogeneous system and a partial solution of (1.1):

(2.3)

$$x_{ij}^k = \sum_{(\tau,\rho)^k \in U^k \setminus U_L^k} x_{\tau\rho}^k \delta_{ij}^k(\tau, \rho) + \left(\tilde{x}_{ij}^k - \sum_{(\tau,\rho)^k \in U^k \setminus U_L^k} \tilde{x}_{\tau\rho}^k \delta_{ij}^k(\tau, \rho) \right), (i, j)^k \in U_L^k;$$
$$x_{\tau\rho}^k \in \mathbb{R}, \quad (\tau, \rho)^k \in U^k \setminus U_L^k,$$

where $\tilde{x}^k = (\tilde{x}_{ij}^k, (i, j)^k \in U^k)$ is a partial solution of the system (1.1) for a fixed $k \in K$; $x_{\tau\rho}^k$ are independent variables corresponding to arcs $(\tau, \rho)^k \in U^k \setminus U_L^k$.

3. Decomposition of the system

We define a set $U_B = \{U_B^k \subseteq U^k \backslash U_L^k, k \in K\}, |U_B| = q + |U_0|$ of bicyclic arcs by selecting $q + |U_0|$ arbitrary arcs from the sets $U^k \backslash U_L^k, k \in K$. We denote $U_N = \{U_N^k, k \in K\}, U_N^k = U^k \backslash (U_L^k \bigcup U_B^k), k \in K$. Thus, $U^k = U_L^k \cup U_B^k \cup U_N^k$, where U_L^k, U_B^k, U_N^k are non-intersecting subsets of arcs.

Let us choose the partial solution of (1.1), for a fixed $k \in K$, such that $\tilde{x}_{\tau\rho}^k = 0$, $(\tau, \rho)^k \in U^k \backslash U_L^k$. Substitution of the general solution (2.3), with a partial solution of the form described above, for each $k \in K$, into (1.2) and (1.3) leads to the following system for finding the unknowns $x_B = (x_{\tau\rho}^k, (\tau, \rho)^k \in U_B^k, k \in K)$, ordered according to an arbitrary numbering $t = t(\tau, \rho)^k, (\tau, \rho)^k \in U_B^k, k \in K, t \in \{1, 2, \ldots, |U_B|\}$:

$$Dx_B = \beta, \tag{3.1}$$

where $D = \binom{D_1}{D_2}$, $D_1 = (\Lambda^{kp}_{\tau\rho}, p = \overline{1,q}, t(\tau,\rho)^k = \overline{1,|U_B|})$, $D_2 = (\delta_{ij}(B^k_{\tau\rho}), \xi(i,j) = \overline{1,|U_0|}, t(\tau,\rho)^k = \overline{1,|U_B|})$, $\beta' = \left(\beta_p, p = \overline{1,q}; \beta_{q+\xi(i,j)}, (i,j) \in U_0\right)'$; $\xi = \xi(i,j)$ is a number of an arc $(i,j) \in U_0$, $\xi \in \{1,2,\ldots,|U_0|\}$.

Here,

$$\Lambda^{kp}_{\tau\rho} = \lambda^{kp}_{\tau\rho} + \sum_{(i,j)^k \in U^k_L} \lambda^{kp}_{ij} \delta^k_{ij}(\tau,\rho), \quad (\tau,\rho)^k \in U^k \backslash U^k_L, \tag{3.2}$$

$$\delta_{ij}(B^k_{\tau\rho}) = \begin{cases} \delta^k_{ij}(\tau,\rho), k \in K_0(i,j) \\ \\ 0, k \notin K_0(i,j) \end{cases}, (i,j) \in U_0, (\tau,\rho)^k \in U^k \setminus U^k_L, k \in K, \tag{3.3}$$

$$\beta_p = A^p - \sum_{k \in K} \sum_{(\tau,\rho)^k \in U^k_N} \Lambda^{kp}_{\tau\rho} x^k_{\tau\rho}, p = \overline{1,q}, \tag{3.4}$$

$$\beta_{q+\xi(i,j)} = A_{ij} - \sum_{k \in K} \sum_{(\tau,\rho)^k \in U^k_N} \delta_{ij}(B^k_{\tau\rho}) x^k_{\tau\rho}, \quad (i,j) \in U_0, \tag{3.5}$$

$$A^p = \alpha_p - \sum_{k \in K} \sum_{(i,j)^k \in U^k_L} \lambda^{kp}_{ij} \, \tilde{x}^k_{ij}, \; p = \overline{1,q}, \tag{3.6}$$

$$A_{ij} = z_{ij} - \sum_{\substack{k \in K_0(i,j), \\ (i,j)^k \in U^k_L}} \tilde{x}^k_{ij}, \quad (i,j) \in U_0. \tag{3.7}$$

Finally, letting $D^{-1} = (\nu_{l,s}; l,s = \overline{1,|U_B|})$, using (2.3) and (3.1), as well as formulas (3.2)-(3.7), we can determine the general solution of (1.1) - (1.3):

$$x^k_{\tau\rho} = \sum_{p=1}^{q} \nu_{t,p}\beta_p + \sum_{(i,j)\in U_0} \nu_{t,q+\xi(i,j)}\beta_{q+\xi(i,j)}, t = t(\tau,\rho)^k, \; (\tau,\rho)^k \in U^k_B, \; k \in K,$$

$$x^k_{ij} = \sum_{(\tau,\rho)^k \in U^k_N} x^k_{\tau\rho}\delta^k_{ij}(\tau,\rho) + \psi^k_{ij} + \tilde{x}^k_{ij}, \; (i,j)^k \in U^k_L, \; k \in K,$$

$$x^k_{\tau\rho} \in \mathbb{R}, \; (\tau,\rho)^k \in U^k_N, \psi^k_{ij} = \sum_{(\tau,\rho)^k \in U^k_B} x^k_{\tau\rho}\delta^k_{ij}(\tau,\rho).$$

References

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